Rutgers University: Algebra Written Qualifying Exam

January 2015: Problem 1 Solution

Exercise. Prove that the group $\mathbb{Q}$ of rationals under addition is a torsion free abelian group, but is not a free abelian group.

Solution.

$(\mathbb{Q}, +)$ is obviously abelian because addition is commutative in $\mathbb{R}$.
A **torsion free** froup is a group where the only element of finite order is the identity.
Suppose $\frac{m}{n} \in (\mathbb{Q}, +)$ has finite order $k \in \mathbb{N}$.

$$\text{Then} \quad \underbrace{\frac{m}{n} + \cdots + \frac{m}{n}}_{k \text{ times}} = 0$$

$$\implies \quad \frac{mk}{n} = 0$$

$$\implies \quad m = 0 \qquad \text{since } k \neq 0 \text{ and } \mathbb{Q} \text{ has no nonzero zero divisors.}$$

Thus, if $\frac{m}{n}$ has finite order, it must be 0, the identity.
$\implies$ $(\mathbb{Q}, +)$ is torsion free.

Prove $(\mathbb{Q}, +)$ is not a free abelian group.
A **free group** is a group with a basis.
Since $(\mathbb{Q}, +)$ is not cyclic, the basis would have **at least** 2 elements, $\frac{a}{b}$ and $\frac{m}{n}$.
But $-an\left(\frac{m}{n}\right) + mb\left(\frac{a}{b}\right) = -am + am = 0$, a contradiction.
Thus, $(\mathbb{Q}, +)$ is not a free abelian group.